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Partitioning a rectangle into small perimeter rectangles

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Abstract

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We show that the way to partition a unit square into k^2+s rectangles, for $s=1$ or $s=-1$, so as to minimize the largest perimeter of the rectangles, is to have $k-1$ rows of k identical rectangles and one row of $k+s$ identical rectangles, with all rectangles having the same perimeter. We also consider the analogous problem for partitioning a rectangle into n rectangles and describe some possible approaches to it.

1. Introduction

Motivated by a certain scheduling problem, Hurwicz [2] raised the following question: for $n \geqslant 1$, how can one partition or cover the unit square with n rectangles (whose edges are parallel to the axes), so as to minimize the largest perimeter among the rectangles?

The same problem was raised by Kasif and Klette [3], who were motivated by a certain data allocation problem in which each rectangle represents a set of tasks performed by one of n parallel processors, and its perimeter corresponds to the memory required by that processor.

In a previous note (see [1]), the authors derived several inequalities that must hold for any partition of the square into rectangles (or polyominoes), and used these to find lower bounds on the maximum perimeter of a rectangle in such a partition or cover.

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An easier version of the partitioning problem, in which it is required that all rectangles have the same area, has been solved completely in [4] and in [5]. The result in this case is obtained by showing that in any partitioning of the unit square into n rectangles, where n is bigger than l^2 and smaller than $(l+1)^2$, there will always be rectangles with a side of length at least $1/l$, as well as rectangles with a side of length at most $1/(l+1)$.

Without the requirement on the equal areas the problem seems much more difficult. Anderson had conjectured that the best construction, for $n = k^2 + sj$, with $s = 1$ or $s = -1$, and $0 \leq j \leq k$ has j rows each consisting of $k + s$ identical rectangles and $k - j$ rows of k identical rectangles, so that all rectangles have the same perimeter.

The lower bounds of [1] mentioned above are achieved by this construction for $j = 0$ and $j = k$, so that the perimeters achieved by it are optimal in these cases.

In the present note we show that this construction is also best possible for $j = 1$ in the partition case. We also define a generalization of the problem, namely that in which we have a 1 by x rectangle instead of the unit square, and give some results for this problem. Finally we discuss some ideas that might be used to solve the original problem in general.

2. Partitioning the unit square into n nearly disjoint squares

We first discuss the general partitioning problem and obtain two results concerning it. We then apply these results to prove the Anderson Conjecture for $j = 1$ (i.e., for $n = k^2 + 1$ or $n = k^2 - 1$) in the partitioning case.

Theorem 1. *Let $n \geq 1$ be an integer and suppose the unit square is partitioned into n rectangles. Suppose these rectangles can be partitioned into blocks so that the sum of the lengths of the members of each block in the direction of one of the axes is* 1. *Then the maximum rectangle perimeter is at least that given by the Anderson construction for n.*

Proof. We accomplish this proof in three steps; we first show that, with a given maximum perimeter p, the maximum area in the rectangles in any one set of total length one is achieved when all the rectangles in it have the same length and maximum perimeter. Then we note that the total area in the union of such sets of rectangles is maximized when the cardinalities of the sets in the partition differ by at most 1. Since the rectangles partition the square the sum of their areas is 1 and this supplies a lower bound for p in terms of the number of blocks of rectangles we have. We finally observe that the lower bound is smallest when the number k of blocks in the partition satisfies $|n - k^2| \leq k$. This case corresponds to the Anderson Conjecture and is realized by its construction for n rectangles.

The basis for our argument is given in the following lemma, which will also be useful later.

Lemma 1. *If the sum of the lengths of k rectangles is x, and their maximum perimeter is p, then the sum of their areas is at most $x(p/2 - x/k)$. This bound is realized when all the rectangles are identical.*

Proof. Let the length of the jth rectangle be $l_j = x/k + d_j$ and let its perimeter be $p - e_j$. Then its area is $(x/k + d_j)(p/2 - x/k - d_j) - l_j e_j/2$. If we sum this over all k rectangles, and use the fact that the sum of the d_j is zero, we conclude that the total area of all k rectangles is $x(p/2 - x/k) - \sum_j (d_j^2 + l_j e_j/2)$.

This expression is maximized when the sum in it vanishes, completing the proof of the lemma. □

Returning to the proof of Theorem 1 suppose that our rectangles are partitioned into m blocks, the ith of cardinality k_i, so that the lengths in each block sum to 1. We must then have, by Lemma 1, that the total area of our square, 1, is at most the sum of $(p/2 - 1/k)$ over the blocks, i.e.,

$$1 \leq \frac{mp}{2} - \sum_{i=1}^{m} \frac{1}{k_i}, \tag{1}$$

where k_i is the number of rectangles in block number i, and hence

$$\sum_{i=1}^{m} k_i = n. \tag{2}$$

For any fixed m, the right hand side of inequality (1), subject to the constraint (2) is maximized when the numbers k_i are as equal as possible, i.e., all of them are within 1 of n/m. We may thus obtain an explicit lower bound on p as a function of m and n. If there are m blocks and $n = ma + b$, where $0 \leq b \leq m$ then $1 \leq mp/2 - b/(a+1) - (m-b)/a$ and hence

$$p \geq \frac{2}{m} + \frac{2}{a} - \frac{2b}{ma(a+1)}. \tag{3}$$

Each of these bounds can be realized, by choosing b rows of $a + 1$ identical rectangles and $m - b$ rows of a identical rectangles, where all rectangles have the same perimeter. The value of m that makes this bound least therefore gives the achievable minimum among partitions into rectangles with the property assumed in this theorem. For $n = k^2 + j$, where $0 \leq j \leq k$ we can set $b = j$, $a = k$ and $m = k$, whereas for $n = k^2 - j$, where $0 \leq j \leq k$ we can set $b = k - j$, $a = k - 1$ and $m = k$. Moreover, it is not too difficult to check that these choices of m give the least possible values for the right-hand side of inequality (3) for the corresponding value of n.

It follows that the optimum solutions for the partition problem among those having the properties assumed in Theorem 1 are those of the Anderson Conjecture. □

Given a partition of a rectangle into rectangles let us define a (horizontal or vertical) *cross section* to be the set of all rectangles that have a non-empty intersection with a fixed line parallel to the horizontal or vertical axis. (We assume that all our rectangles are semi-closed, i.e., they contain their left side and their bottom, but do not contain their right side and their top; this way any point of the partitioned rectangle belongs to exactly one of the rectangles partitioning it.)

Our second result concerning the general partitioning problem is the following somewhat technical but useful statement.

Lemma 2. *Suppose we have a partition of the unit square into rectangles in which the rectangles cannot be partitioned into disjoint blocks such that the sum of the vertical lengths of the rectangles in each block is* 1. *Then exactly one of the following two possibilities holds.*

(i) *There are horizontal cross sections of at least two distinct cardinalities.*

(ii) *There is an integer k such that each horizontal cross section is of cardinality k and there are two sets A and B of rectangles, each of total horizontal length* 1, *with one or more rectangles in common, such that $|A| \leq k-1$, $|B| \geq k+1$ and $|A|+|B|=2k$.*

Proof. Suppose (i) does not hold, and let k be the cardinality of each horizontal cross section. Let us assign a *rank* to each rectangle in our partition as follows. Let R be a rectangle in the partition. If every horizontal cross section that intersects R has R in position j from the left then assign rank j to R. Otherwise assign rank infinity to R. (It is convenient to start counting the possible positions from 0.) We claim that at least one rectangle is assigned an infinite rank. Indeed, otherwise we can define B_i to be the set of all rectangles of rank i, for $1 \leq i \leq k$. Clearly $B_1 B_2, \ldots, B_k$ form a partition of the rectangles into disjoint blocks. Moreover, the sum of the vertical lengths of the rectangles in each block B_i is 1. This can be proved by induction on i. It is clearly true for $i=1$, as the left sides of the rectangles in B_1 partition the left edge of the unit square. Assuming it holds for $i-1$, it clearly holds for i, since the union of the left sides of the members of B_i coincides with the union of the right sides of the members of B_{i-1}. Thus, if every rectangle is assigned a finite rank the rectangles can be partitioned into blocks in a way that violates the assumptions of the theorem. Hence there is a rectangle with infinite rank, as claimed.

Let Y be such a rectangle. Then Y appears in two cross sections C_1 and C_2 in different positions, say j_1 and j_2, respectively, from the left. Without loss of generality assume $j_1 \leq j_2$. Let A be the set of j_1 rectangles in C_1 to the left of Y, Y

itself and the set of $k-j_2-1$ rectangles in C_2 to the right of Y. Similarly, Let B be the set of j_2 rectangles in C_2 to the left of Y, Y itself and the set of $k-j_1-1$ rectangles in C_1 to the right of Y. Clearly A and B satisfy the assertion in part (ii) of the theorem. Therefore, if (i) does not hold then (ii) holds, completing the proof. □

Next we apply Theorem 1 and Lemma 2 to show that the Anderson Conjecture holds for $n=k^2+1$ and for $n=k^2-1$.

Theorem 2. *When* $n=k^2+1$ *or* $n=k^2-1$, *the Anderson construction has smallest possible maximum perimeter.*

Proof. We prove the case $n=k^2-1$. The proof for $n=k^2+1$ is analogous. Given a partition of the unit square into $n=k^2-1$ rectangles we have to show that the maximum perimeter is at least $4/k+2/k^2(k-1)$. By Theorem 1, Lemma 2 (and by the statement obtained from Lemma 2 by interchanging 'vertical' and 'horizontal') the desired result holds unless either

(a) the assertion of Lemma 2 (ii) holds (for horizontal or vertical cross sections), or

(b) there are cross sections of two different cardinalities in both the horizontal and the vertical directions.

Put $q=4/k+2/k^2(k-1)$. We complete the proof by showing that if the maximum perimeter is no larger than q, and either (a) or (b) holds, then the total area of the rectangles is strictly less than 1. Thus, these cases are impossible and the maximum perimeter is at least q, as needed.

Clearly, if no perimeter is greater than q, then the area of each rectangle is at most $(q/4)^2$, where equality holds iff the rectangle is a square of maximum perimeter. Note that

$$(k^2-1)\left(\frac{q}{4}\right)^2-1=\frac{1}{k^3}+\frac{k+1}{4k^4(k-1)}$$

and hence it suffices to show that in cases (a) and (b) the total 'area loss' arising from the rectangles which are not squares of maximum perimeter exceeds $1/k^3+(k+1)/4k^4(k-1)$.

We consider the two cases (a) and (b) separately.

Case (a): Let A and B be the two sets of rectangles, each of total length 1 (horizontal or vertical), where $|A|+|B|=2k$, $|A\cap B|=j\geq 1$ and $|A|<|B|$. Suppose $|A|=k-1-l$, $|B|=k+1+l$, where $l\geq 0$, and let x be the total length of the rectangles in $A\cap B$. By applying Lemma 1 to the rectangles in $A\setminus B$ we conclude that their total area is at most $(1-x)(q/2-(1-x)/(k-1-j-l))$. Similarly, the total area of the rectangles in $B\setminus A$ is at most

$$(1-x)(q/2-(1-x)/(k+1-j+l)).$$

Therefore, the total area of the $2k-2j$ rectangles in $(A\setminus B)\cup(B\setminus A)$ is at most

$$S=(1-x)\left[q-(1-x)\left(\frac{1}{k-1-j-l}+\frac{1}{k+1-j+l}\right)\right].$$

By multiplying the two sides of the equality by $1/(k-1-j-l)+1/(k+1-j+l)$ and by using the fact that the maximum of the function $f(z)=z(q-z)$ is $q^2/4$, we conclude that

$$\left(\frac{1}{k-1-j-l}+\frac{1}{k+1-j+l}\right)S\leqslant q^2/4$$

and hence that

$$S\leqslant\frac{q^2}{4}\frac{(k-1-j-l)(k+1-j+l)}{2k-2j}\leqslant\frac{q^2}{4}\frac{(k-1-j)(k+1-j)}{2k-2j}$$
$$=\left(\frac{q}{4}\right)^2\left(2k-2j-\frac{4}{2k-2j}\right)\leqslant\left(\frac{q}{4}\right)^2(2k-2j)-\frac{2}{k-1}\left(\frac{q}{4}\right)^2.$$

It follows that the area loss arising from these rectangles i.e., the difference between the total area of $2k-2j$ squares of perimeter q and the total area of the rectangles in $(B\setminus A)\cup(A\setminus B)$ is at least

$$\frac{2}{k-1}(q/4)^2=\frac{2}{k^2(k-1)}+\frac{2}{k^3(k-1)^2}+\frac{1}{2k^4(k-1)^3}.$$

This quantity is greater than $1/k^3+(k+1)/4k^4(k-1)$ for all $k\geqslant 2$ and hence in this case the total area is strictly smaller than 1, completing the proof in Case (a).

Case (b): Observe, first, that in this case $k\geqslant 3$. Let A be a horizontal cross section of cardinality other than k, and let B be a vertical cross section of cardinality other than k. Note that if a rectangle of perimeter at most q has an edge of length $q/4+\epsilon$ (where ϵ is either positive of negative), then the area loss arising from this rectangle, i.e., the difference between $(q/4)^2$ and its area is at least ϵ^2. Let R_1 be the unique rectangle in $A\cap B$, let $q/4+\epsilon_1$ and $q/4+\delta_1$ be its width and height, respectively, and assume, without loss of generality, that $|\epsilon_1|\geqslant|\delta_1|$. Let $R_2,\ldots,R_l$ be the other rectangles in A and let $q/4+\epsilon_i$ be the width of R_i. Similarly, let $T_2,\ldots,T_s$ be the other rectangles in B, and let $q/4+\delta_j$ be the height of T_j. Then both l and s differ than k and the total area loss arising from the rectangles in $A\cup B$ is at least

$$L=\epsilon_1^2+\sum_{i=2}^{l}\epsilon_i^2+\sum_{j=2}^{s}\delta_j^2\geqslant\tfrac12\epsilon_1^2+\sum_{i=2}^{l}\epsilon_i^2+\tfrac12\delta_1^2+\sum_{j=2}^{s}\delta 2j$$
$$=((\tfrac12\epsilon_1)^2+(\tfrac12\epsilon_1)^2+\epsilon_2^2+\cdots+\epsilon_l^2)+((\tfrac12\delta_1)^2+(\tfrac12\delta_1)^2+\delta_2^2+\cdots+\delta_s^2)$$
$$\geqslant\frac{(\tfrac12\epsilon_1+\tfrac12\epsilon_1+\epsilon_2+\cdots+\epsilon_l)^2}{l+1}+\frac{(\tfrac12\delta_1+\tfrac12\delta_1+\delta_2+\cdots+\delta_s)^2}{s+1}$$
$$=\frac{(1-lq/4)^2}{l+1}+\frac{(1-sq/4)^2}{s+1}.$$

(Here we used the fact that for m reals $x_1, \ldots, x_m$, $\sum_{i=1}^m x_i^2 \geqslant (\sum_{i=1}^m x_i)^2/m$ and the fact that $\sum_{i=1}^l (q/4 + \epsilon_i) = \sum_{j=1}^s (q/4 + \delta_j) = 1$).

Our objective is to show that $L > 1/k^3 + (k+1)/4k^4(k-1)$, where $l, s \neq k$. It clearly suffices to check that for each $l \neq k$

$$\frac{(1 - lq/4)^2}{l+1} > \frac{1}{2}\left(\frac{1}{k^3} + \frac{k+1}{4k^4(k-1)}\right). \tag{4}$$

For $l = k - 1$ this inequality is

$$\left(1 - (k-1)\left(\frac{1}{k} + \frac{1}{2k^2(k-1)}\right)\right)^2 > \frac{k}{2}\left(\frac{1}{k^3} + \frac{k+1}{4k^4(k-1)}\right) \tag{5}$$

and for $l = k + 1$ it is

$$\left((k+1)\left(\frac{1}{k} + \frac{1}{2k^2(k-1)} - 1\right)\right)^2 > \frac{k+2}{2}\left(\frac{1}{k^3} + \frac{k+1}{4k^4(k-1)}\right). \tag{6}$$

Moreover, the function $f(l) = (1 - lq/4)^2/(l+1)$ is decreasing for all $l \leqslant k - 1$ and is increasing for all $l \geqslant k+1$ (since for $l \geqslant k+1$, $lq/4 > 1$ and hence the numerator increases by a factor of more than $((l+1)/l)^2$ when we replace l by $l+1$, and the denominator only increases by a factor of $(l+2)/(l+1)$.) Therefore, for each fixed k, the left hand side of (4) attains its minimum for $l < k$ at $l = k - 1$ and its minimum for $l > k$ at $l = k + 1$, and hence (4) follows from (5) and (6).

Inequality (5) is equivalent to

$$2\left(1 - \frac{1}{2k}\right)^2 > 1 + \frac{k+1}{4k(k-1)} \quad \text{or} \quad \frac{k-2}{k} + \frac{1}{2k^2} > \frac{k+1}{4k(k-1)}$$

and this is trivially true for all $k \geqslant 3$.

Inequality (6) is equivalent to

$$\frac{2}{k^2} + \frac{2(k+1)}{k^3(k-1)} + \frac{2(k+1)^2}{4k^4(k-1)^2} > \frac{k+2}{k^3} + \frac{(k+1)(k+2)}{4k^4(k-1)},$$

which is trivially true since the first term on the left is bigger than the first term on the right and the second term on the left is bigger than the second term on the right for all $k > 2$. This completes the proof in Case (b) and hence implies the assertion of Theorem 2. □

3. The nonsquare problem

Suppose we wish to partition or cover the 1 by x rectangle with n rectangles so as to minimize the maximum perimeter of the covering rectangles.

There are three approaches that give results on these questions. One may mimic the argument used by the authors in [1] for the square case. One may also follow the lines of the argument given in Section 2. Finally one can apply the results of the square case to the rectangle problem.

One can conjecture that the smallest possible perimeter is what follows from the natural generalization of the Anderson Conjecture.

Conjecture. The optimum configuration consists of at most two kinds of rectangles; these can be arranged in rows of cardinality k and $k+1$, all rectangles in any row being congruent; the number of rows (and hence the cardinality k), as well as the orientation of the rows (either in the direction of length 1 or in that of length x) being chosen to minimize the implied perimeter.

Theorem 1 here immediately generalizes, and in consequence one may deduce that this conjecture is correct whenever (in the partition problem) all rows but one have the same cardinality, the rectangles in the other rows are close enough to being squares, and the rows run in the longer direction in the overall rectangle. One can deduce precise bounds on the parameters for which this result can be proven, but we will not do so here.

The arguments given in [1] for the partition or cover problem also generalize, giving rise to similar constraints, obtained by integrating properly defined functions on the boundary of the large rectangle, as done in the square case. This would give rise to a similar linear program. Again, in appropriate ranges of x, for n for which the conjectured solution has only one kind of rectangle, the bound obtainable from this linear program is exact, both for covering and partitioning. We omit the details.

One can also obtain conclusions of the 1 by x rectangle problem for certain x and n by observing that any part of an optimum configuration must itself be optimum. Thus, for example, we already know that for $n = k(k+1)$ the regular k by $k+1$ arrangement of rectangles in the square is optimal for covering or partitioning. Therefore, if we omit one of the $k+1$ rows, we find that the regular k by k arrangement must be optimal for either problem when $x = (k+1)/k$. One can similarly deduce from our result here that the regular k by j arrangement is optimal for the partition problem when

$$x = \frac{j(2k^2 - 2k + 1)}{2k^2(k-1)} \quad \text{for } j < k.$$

The general conjecture is that the optimum configuration is semiregular, in that it contains only two kinds of rectangles. The approaches to this problem used heretofor have involved obtaining constraints that lead to this conclusion in particular cases. It would be very nice if one could find a direct proof of this kind of semiregularity in general. Given this, computation of the exact solution becomes a straightforward optimization problem. Unfortunately, we have not been able to find such a proof so far.

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